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1979 J. Phys. A: Math. Gen. 12 L67

(http://iopscience.iop.org/0305-4470/12/4/001)

J. Phys. A: Math. Gen., Vol. 12, No. 4, 1979. Printed in Great Britain

LETTER TO THE EDITOR

Bäcklund transformations of axially symmetric stationary gravitational fields

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Received 20 October 1978

Abstract. A generation theorem for solutions of Einstein's equations is presented. It consists mainly of algebraic steps. With its aid, one obtains from an 'old' solution (e.g. from the Minkowski space) 'new' solutions with an arbitrary number of constants. The method of repeated application of potential and coordinate transformations considered by Geroch and Kinnerley is included.

Non-linear phenomena are being increasingly investigated in many areas of physics. In order to develop their imaginations in the field of non-linearity, physicists are highly interested in the systematic generation of exact solutions of non-linear partial differential equations. A fair example is the manner and method in which Bäcklund transformations provide us with solutions of the sine–Gordon equation, which applies to a number of physical problems (model quantum theories, dislocation theory, ultrashort pulses, etc). If one has a solution of this partial differential equation, the only work to be done in obtaining new solutions is to solve an ordinary (total) Riccati differential equation. After that, new solutions with an arbitrary number of constants can be generated by algebraic operations alone.

We have found that axially symmetric stationary vacuum solutions of Einstein's equations can be generated in just the same way. The system of first-order differential equations

$$M_{i,2} = C_i^{kl} M_k N_l; \qquad N_{i,1} = C_i^{kl} N_k M_l \tag{1}$$

for the unknown functions $M_i(x^1, x^2)$ and $N_i(x^1, x^2)$ contains all solutions of Einstein's vacuum field equations with the said symmetries. To show this, put $i = 3$ and choose as non-vanishing coefficients

$$\begin{aligned} &C_1{}^{11} = C_2{}^{22} = C_3{}^{33} = -C_1{}^{12} = -C_2{}^{21} = -1; \\ &C_1{}^{32} = C_1{}^{13} = C_2{}^{31} = C_2{}^{23} = -\tfrac{1}{2}. \end{aligned} \tag{2}$$

Then the connection between the M's and N's and the components of the metric tensor is given by

$$M_1 = \bar{N}_1 = f_{,1}(f+\bar{f})^{-1}; \qquad M_2 = \bar{N}_2 = \overline{f_{,2}}(f+\bar{f})^{-1}; \qquad M_3 = \bar{N}_3 = V_{,1} V^{-1} \tag{3}$$

where f is the complex gravitational potential and V, satisfying the potential equation $V_{,1,2} = 0$, is the root of the azimuthal metric component $g\phi\phi$. ρ and z in $x^1 \equiv \rho + \mathrm{i}z = \overline{x^2}$ are two-dimensional polar coordinates. A bar denotes complex conjugated quantities.

For $i=2$ and non-vanishing coefficients $C_1{}^{11}=C_2{}^{22}=C_3{}^{33}=-C_1{}^{12}=-C_2{}^{21}=-1$ the system (1) is equivalent to the sinh–Gordon equation.

From a known solution $\mathring{M}_i$, $\mathring{N}_i$ of equations (1) and (2) one obtains new solutions M_i, N_i by the following invariance operations I_1 and S:

$$I_1\colon M_i=\alpha_i{}^k\mathring{M}_k, \qquad N_i=\overset{-1}{\alpha}{}_i{}^k\mathring{N}_k, \qquad (\alpha_i{}^k)=\begin{pmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma\end{pmatrix}, \qquad \alpha\beta=\gamma \tag{4}$$

$$S\colon M_i=\mu_i{}^k\mathring{M}_k, \qquad N_i=\nu_i{}^k\mathring{N}_k, \qquad (\mu_i{}^k)=\begin{pmatrix}0 & -1 & \frac{1}{2}\\ -1 & 0 & \frac{1}{2}\\ 0 & 0 & 1\end{pmatrix},$$

$$(\nu_i{}^k)=\begin{pmatrix}-1 & 0 & \frac{1}{2}\\ 0 & -1 & \frac{1}{2}\\ 0 & 0 & 1\end{pmatrix} \tag{5}$$

where $\overset{-1}{\alpha}{}_i{}^k$ is the inverse of $\alpha_i{}^k$. α and γ are solutions of the total Riccati equations

$$\mathrm{d}\gamma=(\gamma^2-\gamma)\mathring{M}_3\,\mathrm{d}x^1+(\gamma-1)\mathring{N}_3\,\mathrm{d}x^2 \tag{6}$$

$$\begin{aligned}\mathrm{d}(\alpha\gamma^{-1/2})=&[-\gamma^{1/2}\mathring{M}_2+(\alpha\gamma^{-1/2})(\mathring{M}_2-\mathring{M}_1)+(\alpha\gamma^{-1/2})^2\mathring{M}_1\gamma^{1/2}]\,\mathrm{d}x^1\\ &+[-\gamma^{-1/2}\mathring{N}_1+(\alpha\gamma^{-1/2})(\mathring{N}_1-\mathring{N}_2)+\alpha\gamma^{-1/2})^2\mathring{N}_2\gamma^{-1/2}]\,\mathrm{d}x^2.\end{aligned} \tag{7}$$

The integral of (6) is easy to find. General integrals of (7) are available (e.g.) in all cases in which the field equations for $\mathring{M}_i$, $\mathring{N}_i$ reduce to three-dimensional potential equations (Weyl class: $\mathring{M}_1=\mathring{M}_2$; Papapetrou class; van Stockum class: $\mathring{M}_2=\mathring{N}_1=0$). The transformation (4) has the interesting property of leaving invariant the Cosgrove–Tomimatsu–Sato class, which can be obtained from (1) and (2) by a product *ansatz*.

We are interested in product transformations $\ldots I_1SI_1SI_1$. Therefore we consider the free product P of the transformation groups I_1 and $I_2=SI_1S$, which are Bäcklund transformations. I_1 and I_2 respectively are generalisations of an invariance group $I=I_1|\gamma=1$ discovered by Kramer and Neugebauer (1968) and of the Matzner–Misner group $K=I_2|\gamma=1$ corresponding to a coordinate transformation in space–time. For $\gamma=1$ the free product P therefore contains the infinite-parameter group which was investigated by Geroch (1971, 1972), Kinnersley (1977) and Kinnersley and Chitre (1977) by a rather complicated method.

Our main result concerning the representation of P is that having integrated equations (6) and (7) one obtains all elements of P (all product matrices) by algebraic operations alone. This result is based on a commutation theorem which states that, for a given initial solution $\mathring{M}_i$, $\mathring{N}_i$ and an arbitrary Bäcklund transformation $\mathring{M}_i, \mathring{N}_i \xrightarrow{I_1} M_i, N_i$ of it, there are always three Bäcklund transformations

$$M_i, N_i \xrightarrow{I_2'} M_i', N_i'; \qquad M_i', N_i' \xrightarrow{I_1''} M_i'', N_i''; \qquad M_i'', N_i'' \xrightarrow{I_2'''} M_i''', N_i'''$$

leading back to the initial solution

$$\mathring{M}_i, \mathring{N}_i \xrightarrow{I_1} M_i, \qquad N_i \xrightarrow{I_2'} M_i', \qquad N_i' \xrightarrow{I_1''} M_i'', \qquad N_i'' \xrightarrow{I_2'''} \mathring{M}_i, \mathring{N}_i. \tag{8}$$

The corresponding α's and γ's are given by

$$\gamma' = \gamma^{-1}, \qquad \alpha' = (\alpha-\gamma)(\alpha-1)^{-1}\gamma^{-1}; \qquad \gamma'' = \gamma, \qquad \alpha'' = \gamma\alpha^{-1};$$
$$\gamma''' = \gamma^{-1}, \qquad \alpha''' = (\alpha-1)(\alpha-\gamma)^{-1}. \tag{9}$$

In order to calculate the M's and N's one has to perform matrix multiplications (4) and (5). A graph corresponding to the commutation theorem (8) is shown in figure 1. Each corner point corresponds to a solution. Double rulings indicate transformations I_1 and single lines transformations I_2. Composing and deforming graphs we are led to a network like figure 2 which is in general use within the theory of Bäcklund transformations of the sine–(sinh-)Gordon equation. It contains the whole procedure. As a first step one has to solve equations (6) and (7). This gives the initial double rulings $1, 2, 3 \ldots$, where the figures indicate different integration constants in the solution of (6) and (7). From the initial transformations, their inverse transformations are known. Then using (9) all the other lines (transformations) can be calculated algebraically. Following the arrows one obtains the general product transformations I_2I_1, $I_1I_2I_1$, $I_2I_1I_2I_1$, and so on and with them solutions with an increasing number of constants. As an example, the α's and γ's of the second step read as follows:

$$\overset{(2)}{\gamma} = \gamma_2/\gamma_1, \qquad \overset{(2)}{\alpha} = (\gamma_1\alpha_2 - \alpha_1\gamma_2)/\gamma_1(\alpha_2-\alpha_1) \tag{10}$$

where (γ_1, α_1) and (γ_2, α_2) are solutions of the same differential equations (6) and (7) with different integration constants. A detailed analysis of the group P will be published elsewhere. The repeated application of the invariance transformations I and K to a given solution $\mathring{M}_i$, $\mathring{N}_i$ is a special case of our procedure. In this case all the γ's are equal to 1 and the $\overset{(n)}{\alpha}$ $(n>1)$ take the form 0/0. The use of the Bernoulli–l' Hôpital

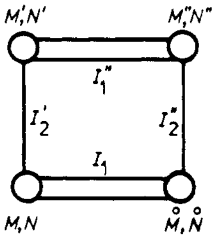

Figure 1. Graph of the commutation theorem. Each corner point corresponds to a solution.

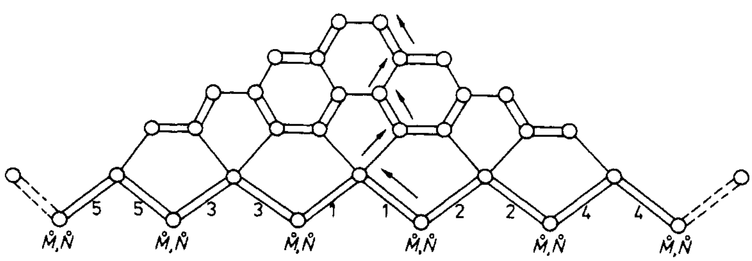

Figure 2. Repeated Bäcklund transformations. The figures indicate transformations of the same kind with different integration constants. In order to obtain solutions with an increasing number of constants one has to calculate the lines denoted by arrows.

theorem leads to derivatives with respect to the integration constants. This fact is well known from sine–(sinh–)Gordon theory.

What about the generation of physically interesting solutions? A stimulating example has been given by Herlt (1979), who was able to generate the Kerr solution from a complex van Stockum solution. (In order to reproduce his result by our present method one has to calculate three arrows in figure 2 and to choose special integration constants.) Probably the presented method of Bäcklund transformations enables us to construct all axially symmetric stationary vacuum solutions. This conjecture is based on the fact that equations (6) and (7) can be transformed into a linear eigenvalue problem conjugated with the non-linear equations (1), and Bäcklund transformations provide us with a solution of this problem. In this way non-linear evolution equations like the sine–Gordon equation are solved (Lax 1968, Ablowitz *et al* 1974).

The Bäcklund transformations have an interesting geometrical meaning. It can be shown that each axisymmetric stationary vacuum field corresponds to a minimal surface on the hyberbolic paraboloid $x+y+u^2+v^2-w^2=0$ embedded in a five-dimensional pseudo-Euclidian space. Therefore our Bäcklund transformations map minimal surfaces into minimal surfaces.

I am greatly indebted to my colleagues Drs Herlt, Kramer, Stephani and Wechsung for many interesting discussions.

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